PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON Tuesday, April 9, 2024, ON SECTION 15.1:
THE DOUBLE INTEGRAL OVER A RECTANGLE AS A LIMIT OF DOUBLE RIEMANN SUMS and CALCULATING its Value USING Iterated Integrals

CLASS #23

INTEGRATION OF Functions of Several Variables

We begin with the Double Integral of two variables.

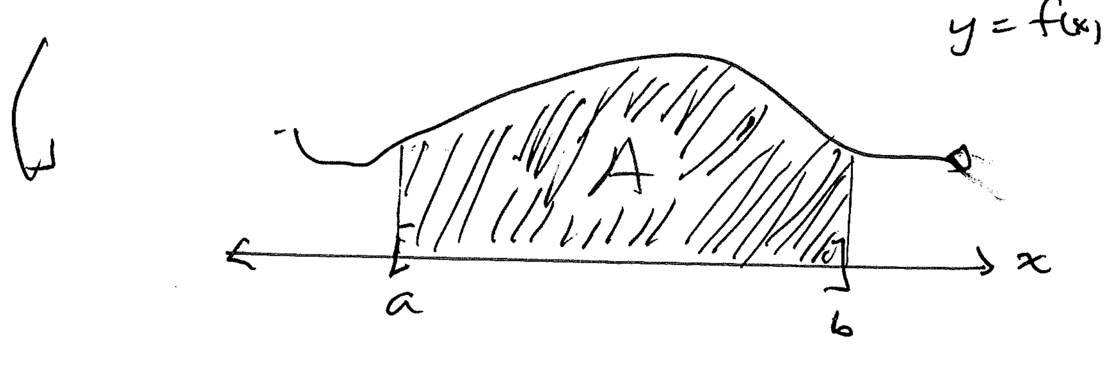
Recall how the Definite Integral $\int_a^b f(x)\,dx$ works.

$$\Delta x = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \Bigg(\underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{A Riemann Sum}} \Bigg).$$

If $f(x) \geq 0$ for all x in $[a,b]$,

Then
$$A = \int_a^b f(x)\,dx = \text{The Area under the curve over } [a,b]$$



15.1 Double Integrals over Rectangles

In much the same way that our attempt to solve the area problem led to the definition of a definite integral, we now seek to find the volume of a solid and in the process we arrive at the definition of a double integral.

Review of the Definite Integral

First let's recall the basic facts concerning definite integrals of functions of a single variable. If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/n$ and we choose sample points x_i^* in these subintervals. Then we form the Riemann sum

$$\boxed{1} \qquad \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

and take the limit of such sums as $n \to \infty$ to obtain the definite integral of f from a to b:

$$\boxed{2} \qquad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

In the special case where $f(x) \geqslant 0$, the Riemann sum can be interpreted as the sum of the areas of the approximating rectangles in Figure 1, and $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b.

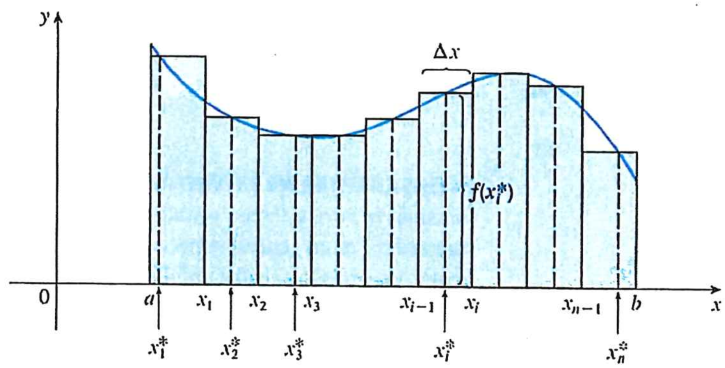


FIGURE 1

Volumes and Double Integrals

In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

and we first suppose that $f(x, y) \geqslant 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}$$

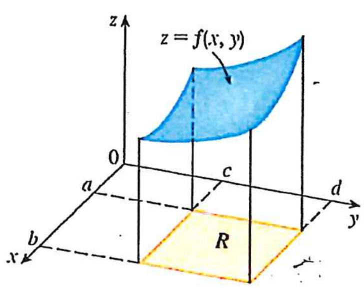


FIGURE 2

(See Figure 2.) Our goal is to find the volume of S.

The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals,

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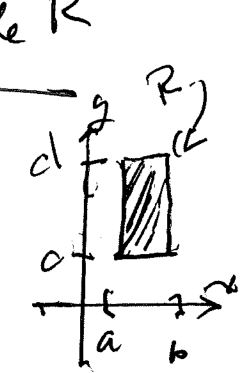
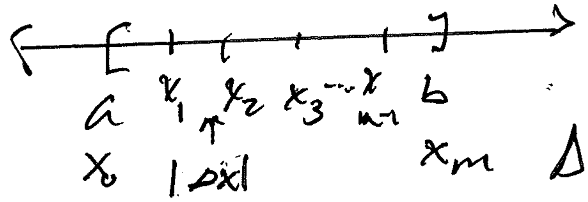
Now, <u>Double Integrals</u> over <u>a</u> <u>Rectangle R</u>

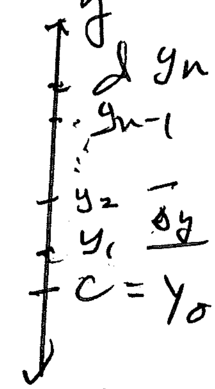


In the xy plane, let R = The Rectangle $[a,b]\times[c,d]$



Subdivide $[a,b]$ into m subintervals

$$\Delta x = \frac{b-a}{m}$$



Subdivide $[c,d]$ into n subintervals

$$\Delta y = \frac{d-c}{n}$$

For each i, $1 \le i \le m$, and for each j, $1 \le j \le n$,

R_{ij} = The subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$

and its area is $\Delta A_{ij} = (\Delta x)(\Delta y) = \Delta A$.

Let $z = f(x,y)$ be a function of 2 variables.

For each i, $1 \le i \le m$ and for each j, $1 \le j \le n$,

<u>select a point (x^*_{ij}, y^*_{ij}) in R_{ij} and</u>

Compute $\displaystyle\sum_{i=1}^{m}\left(\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta A\right)$ The Area of R_{ij}

This number is called a <u>Double Riemann</u> <u>over R</u>.

as in Figure 3, we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$

each with area $\Delta A = \Delta x\, \Delta y$.

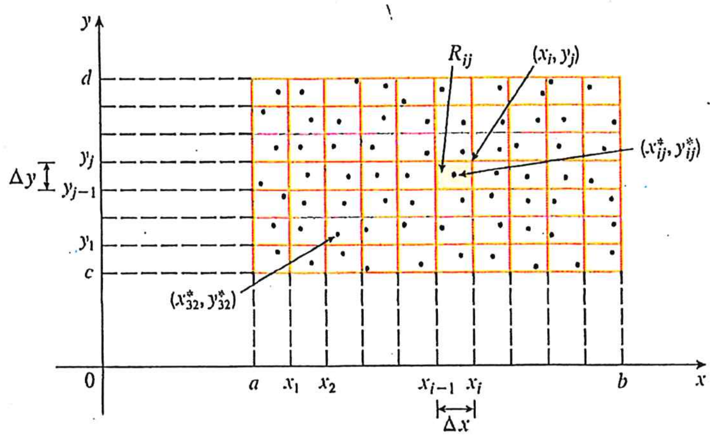


FIGURE 3
Dividing R into subrectangles

If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ as shown in Figure 4. (Compare with Figure 1.) The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

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$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

(See Figure 5.) This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

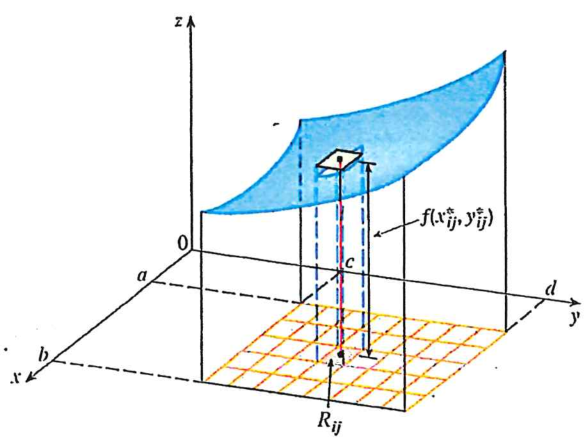

FIGURE 4

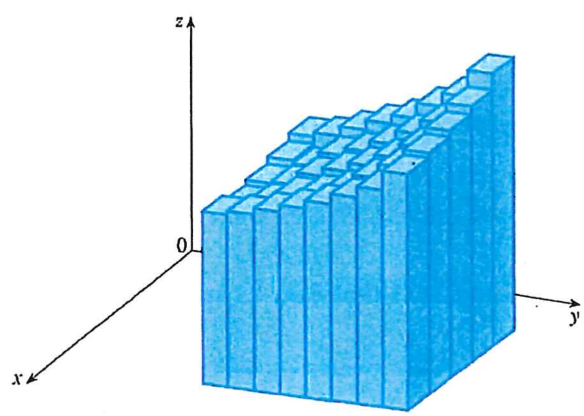

FIGURE 5

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EXAMPLE: Let $f(x,y) = x + 2y + 1$

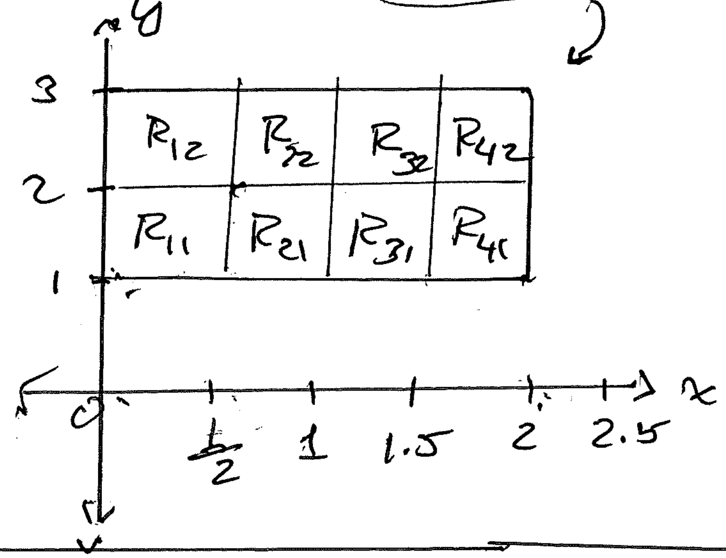
Let $R = [0,2]\times[1,3]$. Let $m = 4$ and $n = 2$

So, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

So, $\Delta y = \frac{3-1}{2} = 1$.

$\Delta A = (\Delta x)(\Delta y) = \frac{1}{2}\times 1 = \frac{1}{2}$

$\frac{1}{2}$ sq. unit



Selecting (x_{ij}^*, y_{ij}^*)

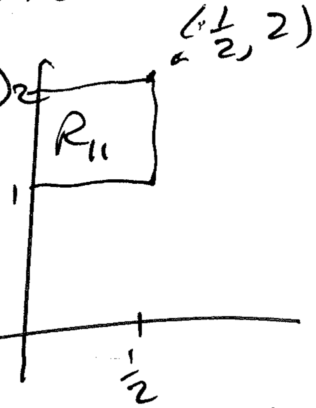
method 1:

The upper right-hand corner method.

$(x_{11}^*, y_{11}^*) = (\frac{1}{2}, 2)$

$(x_{12}^*, y_{12}^*) = (\frac{1}{2}, 3)$

⋮



$(x_{42}^*, y_{42}^*) = (2,3)$

The "4×2" Riemann Sum

Recall $f(x,y) = x + 2y + 1$

is $(\frac{1}{2}+4+1)(\frac{1}{2}) + \cdots + (2+6+1)(\frac{1}{2})$

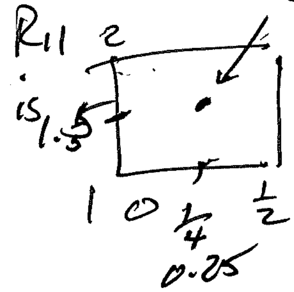
$= \sum_{m=1}^{4}\left(\sum_{n=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A\right) = 24$

method 2: The Midpoint Rule.

(x_{ij}^*, y_{ij}^*) = the midpoint of R_{ij}

$(x_{11}^*, y_{11}^*) = (0.25, 1.5)$

R_{11} is



Using the midpoint rule, the "4×2" Double Riemann Sum is 24.

The sum in Definition 5,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

is called a **double Riemann sum** and is used as an approximation to the value of the double integral. [Notice how similar it is to the Riemann sum in (1) for a function of a single variable.] If f happens to be a *positive* function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume under the graph of f.

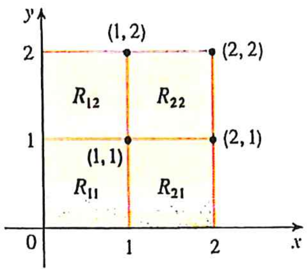


FIGURE 6

EXAMPLE 1 Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

SOLUTION The squares are shown in Figure 6. The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

$$= f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

This is the volume of the approximating rectangular boxes shown in Figure 7. ■

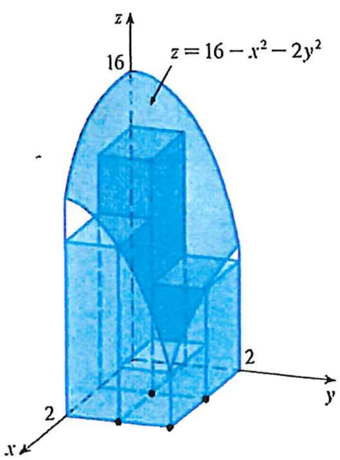


FIGURE 7

We get better approximations to the volume in Example 1 if we increase the number of squares. Figure 8 shows how the columns start to look more like the actual solid and the corresponding approximations become more accurate when we use 16, 64, and 256 squares. In Example 7 we will be able to show that the exact volume is 48.

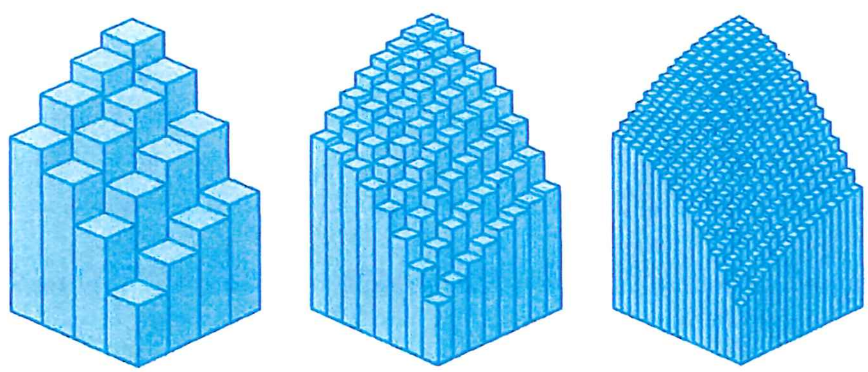


FIGURE 8
The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.

(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

EXAMPLE 2 If $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1 - x^2}\, dA$$

Now, The Double Integral of f

over Rectangle R is the Number

which is

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A\right)\right)$$

In this example, when $f(x,y) = x+2y+1$

and $R = [0,2]\times[1,3]$, The Double Integral

exits and

$$\iint_R (x+2y+1)\,dA = 24$$

How do we determine the Number?

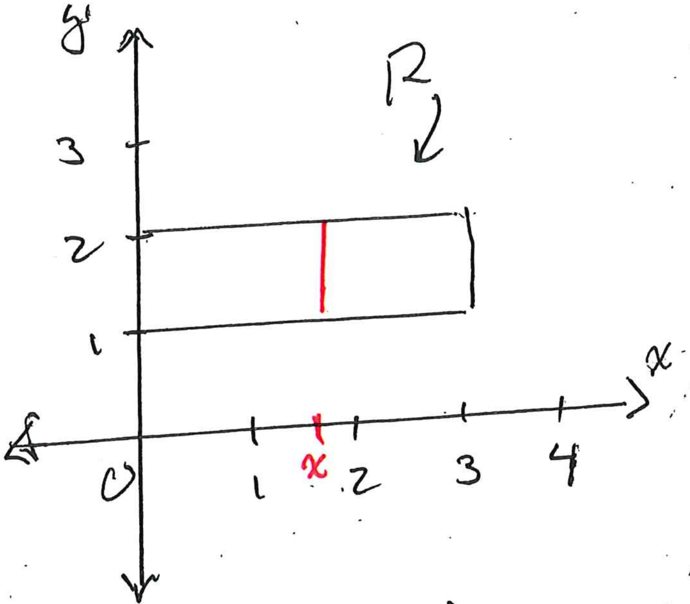
With Iterated Integrals

For example: Let $R = [0,3] \times [1,2]$.

Let $f(x,y) = y^2 + 8xy^3 + x^3$.

How do we find this number,

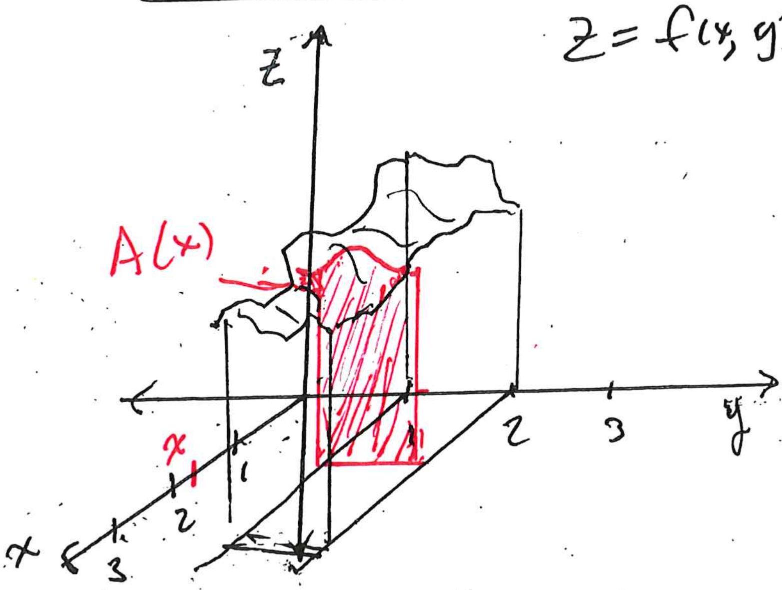
$$\iint_R (y^2 + 8xy^3 + x^3)\,dA \ ?$$



By using Iterated Integrals:
Let x be a fixed # in $[a,b]$, $a \le x \le b$, and, while x is held fixed, let y vary in $[c,d]$, $c \le y \le d$.

Then, we define $g_x(y) = f(x,y)$; $c \le y \le d$.
The function $g_x(y)$ is a function of y for each x, $a \le x \le b$.

$z = f(x,y) = y^2 + 8xy^3 + x^3$



Here we define, for each x, $0 \le x \le 3$,

$g_x(y) = y^2 + 8xy^3 + x^3$.

$g_1(y) = y^2 + 8y^3 + 1$

$g_{1.5}(y) = y^2 + 12y^3 + 3.375$

$g_2(y) = y^2 + 16y^3 + 8$

Fix x at any number, $a \le x \le b$
and define $A(x) = \int_c^d g_x(y)\,dy = \int_1^2 (y^2 + 8xy^3 + x^3)\,dy$

$A(x) = \int_1^2 (y^2 + 8xy^3 + x^3)\,dy = \left(\frac{1}{3}y^3 + 2xy^4 + x^3y\right)\Big|_{y=1}^{y=2}$

$A(x) = \left(\frac{8}{3} + 32x + 2x^3\right) - \left(\frac{1}{3} + 2x + x^3\right) =$

$\boxed{A(x) = x^3 + 30x + \frac{7}{3}}$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy = \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy = 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

Notice that in Example 4 we obtained the same answer whether we integrated with respect to y or x first. In general, it turns out (see Theorem 10) that the two iterated integrals in Equations 8 and 9 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

Theorem 10 is named after the Italian mathematician Guido Fubini (1879–1943), who proved a very general version of this theorem in 1907. But the version for continuous functions was known to the French mathematician Augustin-Louis Cauchy almost a century earlier.

10 Fubini's Theorem If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

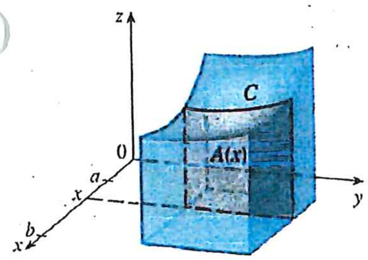


FIGURE 11

The proof of Fubini's Theorem is too difficult to include in this book, but we can at least give an intuitive indication of why it is true for the case where $f(x, y) \geqslant 0$. Recall that if f is positive, then we can interpret the double integral $\iint_R f(x, y) \, dA$ as the volume V of the solid S that lies above R and under the surface $z = f(x, y)$. But we have another formula that we used for volume in Chapter 6, namely,

$$V = \int_a^b A(x) \, dx$$

where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis. From Figure 11 you can see that $A(x)$ is the area under the curve C whose equation is $z = f(x, y)$, where x is held constant and $c \leqslant y \leqslant d$. Therefore

TEC Visual 15.1 illustrates Fubini's Theorem by showing an animation of Figures 11 and 12.

$$A(x) = \int_c^d f(x, y) \, dy$$

and we have

$$\iint_R f(x, y) \, dA = V = \int_a^b A(x) \, dx = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

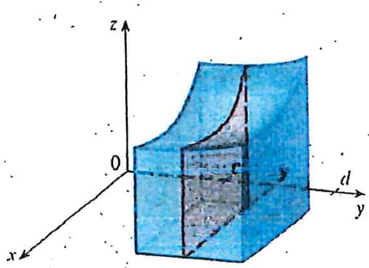


FIGURE 12

A similar argument, using cross-sections perpendicular to the y-axis as in Figure 12, shows that

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

10

$A(x)$ is the cross-sectional area of the Solid Region cut by a vertical plane thouch $(x,0,0)$ and $\perp$ to the x-axis.

<u>Fubini's Theorem</u>:

$$\iint_R f(x,y)\,dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{A(x)} dx$$

$$= \int_c^d \underbrace{\left(\int_a^b f(x,y)\,dx\right)}_{A(y)} dy$$

$$\iint_R (y^2+8xy^3+x^3)\,dA$$

$$= \int_0^3 \left(\int_1^2 (y^2+8xy^3+x^3)\,dy\right) dx$$

$$= \int_0^3 A(x)\,dx, \quad A(x) = x^3+30x+\tfrac{7}{3}.$$

$$= \int_0^3 (x^3+30x+\tfrac{7}{3})\,dx = \dots = 162.25$$

The Volume V of the Solid Region above rectangle R and under the surface graph of $z=f(x,y)=y^2+8xy^3+x^3$ is $V = 162.25$ cubic units.

Problem: Determine the Volume V of the solid region S below the surface graph of $z = 16 - x^2 - y^2$ and above $R = [0,1] \times [0,1]$.

Sol'n: $$V = \iint_R (16 - x^2 - y^2)\, dA$$

$$= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} (16 - x^2 - y^2)\, dy \right) dx$$

$$= \int_{x=0}^{x=1} \left(\left(16y - x^2 y - \tfrac{1}{3} y^3 \right) \Big|_{y=0}^{y=1} \right) dx$$

$$= \int_0^1 \left(\left(16 - x^2 - \tfrac{1}{3} \right) - (0) \right) dx$$

$$= \int_0^1 \left(\tfrac{47}{3} - x^2 \right) dx = \left(\tfrac{47}{3} x - \tfrac{1}{3} x^3 \right) \Big|_0^1$$

$$= \frac{46}{3} = 15\tfrac{1}{3} \text{ cubic units} = V$$

Choosing the Order of Integration
"dy dx" or "dx dy"?

If you end up with difficult integrals, consider switching the order of integration.

Ex. $R = [1,2] \times [0,\pi]$

Determine $\displaystyle\iint_R y \sin(xy)\, dA$

Two Choices

"dy dx"

$$\int_1^2 \int_0^{\pi} y \sin(xy)\, dy\, dx$$

Fix x; and integrate the inside dy

LIKE $\displaystyle\int_0^{\pi} y \sin(3y)\, dy$

if $x=3$.

You would have to use Integration by Parts! Too hard!

"dx dy"

$$\int_0^{\pi} \int_1^2 y \sin(xy)\, dx\, dy$$

Fix y, and integrate the inside dx.

Like $\displaystyle\int_1^2 3 \sin(3x)\, dx$

Much Easier

Choose This! $dx\,dy$